

Odd Hamiltonian Structure for Supersymmetric Sawada - Kotera Equation

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Abstract

We study the supersymmetric $N = 1$ hierarchy connected with the Lax operator of the supersymmetric Sawada-Kotera equation. This operator produces the physical equations as well as the exotic equations with odd time. The odd Bi-Hamiltonian structure for the $N=1$ Supersymmetric Sawada - Kotera equation is defined. The product of the symplectic and implectic Hamiltonian operator gives us the recursion operator. In that way we prove the integrability of the supersymmetric Sawada - Kotera equation in the sense that it has the Bi-Hamiltonian structure. The so called “quadratic” Hamiltonian operator of even order generates the exotic equations while the “cubic” odd Hamiltonian operator generates the physical equations.

1 Introduction

Integrable Hamiltonian systems plays an important place in diverse branches of theoretical physics as exactly solvable models of fundamental physical phenomena ranging from nonlinear hydrodynamics to string theory [1]. There are several different approaches which generalize these models.

One of them is to extend the theory by addition of fermion fields. The first results in this direction can be found in [2, 3, 4, 5, 6, 7]. In many cases, the extension by fermion fields does not guarantee the supersymmetry of the final theory. Therefore this method was called the fermion extension in order to distinguish it from the fully supersymmetric method which was developed later [8, 9, 10, 11].

There are many prescriptions how to embed a given classical model into a fully supersymmetric one. The main idea is simple: in order to obtain an N extended supersymmetry multiplet one has to add to a system of k boson fields kN fermions and $k(N - 1)$ bosons $k = 1, 2, \dots, N = 1, 2, \dots$. Working with this supermultiplet we can apply integrable Hamiltonians methods.

From the soliton theory point of view we can distinguish two important classes of the supersymmetric equations: the non-extended $N = 1$ and extended $N > 1$ cases. The extended case may imply new bosonic equations whose properties need further investigations. Interestingly enough, some typical supersymmetric effects appearing during the supersymmetrizations have important consequences for the soliton theory. Let us mention for instance: the nonuniqueness of the roots of the supersymmetric Lax operator [12], the lack of a bosonic reduction to the classical equations [13], occurrence of non-local conservation laws [14, 15] and existence of odd Hamiltonian structures [16, 17, 18].

The odd Hamiltonian structures first appeared in [19]. It was latter noticed in [20] that in the superspace one can consider both even and odd symplectic structures, with even and odd Poisson brackets, respectively. Recently the odd brackets, also known as antibrackets or Buttin bracket, have been extensively investigated both from the mathematical and the physical point of view. They have drawn some interest in the context of BRST formalism [21], in the supersymmetric quantum mechanics [22], in the classical mechanics [23, 24, 25, 26] and in the gravity theory [27]. Becker and Becker [16] were probably the first who discovered odd Hamiltonian structures in the special non-extended supersymmetric KdV equation. Later these

structures have been discovered in many supersymmetric integrable models [17, 18]. This nonextended supersymmetrization, sometimes called the B extension, are based on the simple transformation of dependent variables describing the classical system, onto superbosons. In that manner the equations are not changed but are rather rewritten in theses new coordinates in the superspace. Such supersymmetrization is however “without the supersymmetry” because the bosonic sector remains the same and does not feel any changes after supersymmetrization.

In this paper we found new unusual features of the supersymmetric models. First we show that recently discovered [28] $N = 1$ supersymmetric extension of the Sawada - Kotera equation, (which is not a B extension) has an odd Bi-Hamiltonian structure. Probably it is a first nontrivial supersymmetric model with an odd Hamiltonian structure and a modified bosonic sector. The conserved quantities are constructed from the fermionic and bosonic fields and disappear in the bosonic or in the fermionic limit. This example shows that from the superintegrability of the whole system one can not conclude an integrability of the bosonic sector.

The second observation is that Lax representation, which generates the supersymmetric Sawada - Kotera equation, generates also exotic equations with odd time. In this paper we treat these equations as subsidiary equations. This allowed us to discovered the Bi-Hamiltonian structure for the supersymmetric Sawada-Kotera equation in which the symplectic and the implectic operators are odd. In that way we proved the integrability of the model in the sense that it possess the Bi-Hamiltonian structure.

The next observation is that the usual even supersymmetric second Hamiltonian operator of the supersymmetric Korteweg - de Vries equation generates also exotic equations. Unfortunately we have been not able to discovered the second Hamiltonian operator for these exotic equations.

We also showed that the odd and the even Hamiltonian operator could be derived systematically using the so called R matrix theory [32, 33, 34, 35, 36, 37].

The paper is organized as follows. In first section we recapitulate known facts about the supersymmetric and the B extensions of Korteweg - de Vries equation and its Bi-Hamiltonian structure. In second section we describe the $N = 1$ supersymmetric Lax representation which produces supersymmetric Sawada - Kotera, the B extension of Kaup - Kupershmidt and the Sawada - Kotera equation. The third section contains our main results. We demonstrate the odd Bi - Hamiltonian structure for the Sawada - Kotera, which has been accidentally discovered from the general considerations and we define the even Hamiltonian operator for the exotic equations. Next we check the validity of the Jacobi identity for our Hamiltonian operator. Finally the proper choice of our Hamiltonians structure is confirmed by computing once more these structures using the so called classical R -matrix theory.

2 Supersymmetric and B extension of the Korteweg - de Vries Equation

Let us consider the Korteweg - de Vries equation in the form

$$u_x = u_{xxx} + 6uu_x \quad (1)$$

where the function u depends on x and t . This equation is usually rewritten in two different forms. The first one utilizes the Bi-Hamiltonian formulation while in the second the Lax representation is used. We have the following Bi-Hamiltonian formulation of the KdV equation

$$u_t = J \frac{\delta H_1}{\delta u} = \partial \frac{\delta H_1}{\delta u} = K \frac{\delta H_2}{\delta u} = (\partial^3 + 2\partial u + 2u\partial) \frac{\delta H_2}{\delta u}, \quad (2)$$

where $H_1 = \frac{1}{2} \int (uu_{xx} + 2u^3) dx$ and $H_2 = \frac{1}{2} \int u^2 dx$ are conserved quantities while $J = \partial_x$ and $P = \partial_{xxx} + 2\partial_x u + 2u\partial_x$ constitute the Hamiltonian operators. The Lax representation is

$$L_t = [L, L_+^{\frac{3}{2}}], \quad (3)$$

where $L = \partial^2 + u$.

We have two possibilities to expand the superfunction Φ as $w(x) + \theta\xi$ where $w(x)$ is an even function and $\xi(x)$ is an odd function in the case of the superboson while in the case of superfermion as $\xi(x) + \theta w(x)$. Hence we can represent the co-vector and the vector fields in two different ways. For the superboson functions we choose the corresponding chart for the vector fields K and co-vector fields Υ as

$$K(\Phi) = K_w + \theta K_\xi \quad \Upsilon(\Phi) = \Upsilon_\xi + \theta \Upsilon_w \quad (4)$$

while for superfermions as

$$K(\Phi) = K_\xi + \theta K_w \quad \mathcal{R}(\Phi) = \mathcal{R}_w - \theta \mathcal{R}_\xi \quad (5)$$

Then the duality between the vector and the co-vector fields becomes the usual one

$$\langle K, \mathcal{R} \rangle = \int K \mathcal{R} dx d\theta = \int (K_w \mathcal{R}_w + K_\xi \mathcal{R}_\xi) dx \quad (6)$$

and the conjugation becomes

$$\langle K, \mathcal{R} \rangle^* = \langle \mathcal{R}, K \rangle = \int (K_w \mathcal{R}_w + K_\xi \mathcal{R}_\xi) dx = \langle K, \mathcal{R} \rangle \quad (7)$$

Let us now try to find the supersymmetric $N = 1$ extension of the KdV equation considering the most general form of the Lax operator

$$L = \partial^2 + \lambda_1 \Phi_1 + \lambda_2 \Phi \mathcal{D} \quad (8)$$

where λ_1 and λ_2 are arbitrary constants, Φ is a superfermionic field of the fractional conformal dimension $\frac{3}{2}$ with the decomposition $\Phi = \xi + \theta u$ where ξ is an odd function and u is an even function. $\mathcal{D} = \partial_\theta + \theta \partial$ is the supersymmetric derivative with the property $\mathcal{D}^2 = \partial$. In the next we use the following notation: Φ_n denotes the value of the action of the operator \mathcal{D}^n on the superfunction Φ , for example $\Phi_0 = \Phi$, $\Phi_1 = (\mathcal{D}\Phi)$, $\Phi_2 = \Phi_x$ while $\mathcal{D}\Psi = \Psi_1(-1)^{[\Psi]}\Psi\mathcal{D}$ where $[\Psi]$ denotes the parity of the superfunction Ψ .

The consistency of Lax representation (3) gives us two solutions for the coefficients λ_1, λ_2

The first solution $\lambda_1 = 0, \lambda_2 = 1$ is the usual well known supersymmetric extension of the KdV equation

$$\Phi_t = \Phi_{xxx} + 3(\Phi\Phi_1)_x \quad (9)$$

In the components this equation takes the form

$$u_t = u_{xxx} + 6uu_x - 3\xi\xi_{xx} \quad (10)$$

$$\xi_t = \xi_{xxx} + 3(u\xi)_x \quad (11)$$

This supersymmetric $N = 1$ KdV equation could be rewritten in the Bi-Hamiltonian form [12]

$$\Phi_t = \Pi \frac{\delta H_1}{\delta \Phi} = (\mathcal{D}\partial^2 + 2\partial\Phi + 2\Phi\partial + \mathcal{D}\Phi\mathcal{D}) \frac{\delta H_1}{\delta \Phi} \quad (12)$$

$$\Omega\Phi_t = (\mathcal{D}\partial^{-1} + \partial^{-1}\Phi\partial^{-1}) \frac{\delta H_2}{\delta \Phi} \quad (13)$$

where

$$H_1 = \int \frac{1}{2} \Phi \Phi_1 dx d\theta = \int \left(\frac{1}{2} (u^2 - \xi\xi_x) \right) \quad (14)$$

$$H_2 = \int \left(\frac{1}{2} \Phi_x \Phi_{1,x} + \Phi \Phi_1 \right)^2 dx d\theta = \int \frac{1}{2} (u_x^2 + 2u^3 + \xi_{xx}\xi_x + 4\xi_x\xi u) dx. \quad (15)$$

Regarding Π operator as a linear map from the co-vector fields to the vector fields the equation (12) could be decomposed into

$$\frac{d}{dt} \begin{pmatrix} u \\ \xi \end{pmatrix} = \begin{pmatrix} \partial^3 + 2\partial u + 2u\partial & \partial\xi + 2\xi\partial \\ 2\partial\xi + \xi\partial & -\partial^2 - u \end{pmatrix} \begin{pmatrix} \frac{\delta H_1}{\delta u} \\ \frac{\delta H_1}{\delta \xi} \end{pmatrix} \quad (16)$$

Now regarding Ω as map from the vector fields to the co-vector fields we translate it to the matrix form

$$\Omega = \begin{pmatrix} \partial^{-1} & \partial^{-1}\xi\partial^{-1} \\ \partial^{-1}\xi\partial^{-1} & -1 - \partial^{-1}u\partial^{-1} \end{pmatrix}. \quad (17)$$

The second solution $\lambda_1 = 1, \lambda_2 = 0$ leads us to the so called B extension of the KdV equation

$$\Phi_t = \Phi_{xxx} + 6\Phi_x\Phi_1 \quad (18)$$

or in the components

$$u_t = u_{xxx} + 6u_x u, \quad (19)$$

$$\xi_t = \xi_{xxx} + 6\xi_x u \quad (20)$$

from which we see that this B extension is “without supersymmetry” because the bosonic sector (19) coincides with the Korteweg - de Vries equation. Moreover the equation (18) also coincides with the usual KdV equation in which we simply replace the function u with the superfunction Φ_1 . These procedure can be considered as the intermediate step between the KdV equation and the potential KdV equation. Indeed applying once more the B-extension to the B-extension of the KdV equation, where now $\Phi = w_1$ and w is a superboson function, we obtain the potential KdV equation.

In general we can carry out such B supersymmetrization to each equation of the form $u_t = F(u)_x$ and easily obtain its Bi-Hamiltonian structure, if such exists. Indeed taking into account the transformation between u and Φ we can obtain the Bi-Hamiltonian structure for the B-extension of the KdV equation

$$\Phi_t = \mathcal{D}^{-1}(\partial^3 + 2(\partial\Phi_1 + \Phi_1\partial))\mathcal{D}^{-1}\frac{\delta H_1}{\delta \Phi} = \mathcal{D}^{-1}\partial\mathcal{D}^{-1}\frac{\delta H_2}{\delta \Phi} = \frac{\delta H_2}{\delta \Phi} \quad (21)$$

where

$$H_1 = \frac{1}{2} \int dx d\theta \Phi \Phi_x = \int dx \xi_x u \quad (22)$$

$$H_2 = \frac{1}{2} \int (\Phi \Phi_{xxx} + 2\Phi_1^3) = \int dx (\xi_{xxx} u + 3\xi_x u^2) \quad (23)$$

. These equation could be translated to the matrix form

$$\frac{d}{dt} \begin{pmatrix} u \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & -\partial^2 - 4u - 2u_x \partial^{-1} \\ \partial^2 + 4u - 2\partial^{-1} u_x & -2\xi_x \partial^{-1} - 2\partial^1 \xi_x \end{pmatrix} \begin{pmatrix} \frac{\delta H_1}{\delta u} \\ \frac{\delta H_1}{\delta \xi} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H_2}{\delta u} \\ \frac{\delta H_2}{\delta \xi} \end{pmatrix} \quad (24)$$

There is a basic difference between the second Hamiltonian operator of the supersymmetric extension and the second Hamiltonian operator of the B-extension of the KdV equation. The first one is an even operator while the second is an odd one. For further investigations we will rather work with the Hamiltonian operator Π then with the Poisson brackets which are connected between themselves as

$$\{F_1, F_2\} = \langle \nabla F_1, \Pi \nabla F_2 \rangle \quad (25)$$

where ∇ denotes the gradient. Using the last formula we associate the components of the second Hamiltonian operator with the abstract Poisson brackets

$$\{\Phi(x, \theta), \Phi(x', \theta')\} = \Pi \delta(x - x')(\theta - \theta') = (\mathcal{D}\partial^2 + 2\partial\Phi(x, \theta) + 2\Phi(x, \theta)\partial + \mathcal{D}\Phi(x, \theta)\mathcal{D}) \delta(x - x')(\theta - \theta') \quad (26)$$

observing that it is even type for the usual $N = 1$ supersymmetric extension of KdV equation while it is odd type

$$\{\Phi(x, \theta), \Phi(x', \theta')\} = \mathcal{D}^{-1}(\partial^3 + 2\partial\Phi_1(x, \theta) + 2\Phi_1(x, \theta)\partial)\mathcal{D}^{-1}\delta(x - x')(\theta - \theta') \quad (27)$$

for B-extension of KdV equation.

In the next we will use the following notation: the even or odd bracket denote that the Poisson bracket for two superfermionic functions f_1, f_2 is superbosonic or superfermionic operator, respectively. On the other side these brackets could be associated with some algebra. In the case of the usual KdV equation these brackets are connected with the Virasoro algebra [30] while in the supersymmetric case with the $N = 1, 2$ supersymmetric Virasoro algebra [9] and for B extension with the $N = 1, 2$ odd extension of the Virasoro algebra [16, 17].

3 Supersymmetric extension of the Sawada - Kotera equation.

In order to generate the Sawada - Kotera equation from the Lax representation let us assume that we have the general third order Lax operator of the form

$$L = \partial^3 + u\partial + \lambda u_x \quad (28)$$

where λ is an arbitrary constant and u is a function of x, t .

This operator generates the whole hierarchy of equations, where the first equation is simply $u_t = u_x$ and the nontrivial ones start from the fifth flow

$$L_t = 9[L, (L^{\frac{5}{3}})_+], \quad (29)$$

$$u_t = (u_{4x} + 5uu_{xx} + 15\lambda(1-\lambda)u_x^2 + \frac{5}{3}u^3)_x \quad (30)$$

only when $\lambda = \frac{1}{2}, 1, 0$.

For $\lambda = \frac{1}{2}$, we have the Kaup - Kupershmidt hierarchy [1, 31] while for $\lambda = 0, 1$ we obtain the Sawada - Kotera hierarchy [1, 31]

The Kaup-Kupershmidt and Sawada-Kotera equations have a similar Bi-Hamiltonian structure where the symplectic operator Γ generates

$$u_t = \Gamma \frac{\delta H}{\delta u} = (\partial^3 + \gamma(\partial_x u + u\partial_x)) \frac{\delta H}{\delta u} \quad (31)$$

$$H = \frac{1}{2} \int dx (uu_{xx} + \beta u^3) \quad (32)$$

where $\gamma = \frac{1}{2}$ and $\beta = \frac{4}{3}$ for Kaup - Kupershmidt equation while for Sawada - Kotera equation we have $\gamma = 2$ and $\beta = \frac{1}{3}$.

The implectic operator for these equations could be obtained from the factorization of the recursion operator [31]

$$\frac{\delta H_1}{\delta u} = \Omega u_t = (\partial^2 + k_1(\partial u + u\partial) + \partial^{-1}(k_1 u_{xx} + k_3 u^2) + (k_2 u_{xx} + k_3 u^2)\partial^{-1})u_t \quad (33)$$

where $k_1 = \frac{5}{2}, k_2 = 1, k_3 = 2$ for the Kaup - Kupershmidt equation while for the Sawada-Kotera equation $k_1 = k_3 = 1, k_2 = \frac{1}{2}$.

Let us now consider in details the supersymmetric $N = 1$ version of the Lax operator (28). The most general form of such generalization is

$$L = \partial^3 + \lambda_1 \Phi_1 \partial + \lambda_2 \Phi_x \mathcal{D} + \lambda_3 \Phi_{1,x} \quad (34)$$

where λ_i are the arbitrary constants and Φ is the superfermionic field of the fractional conformal dimension $\frac{3}{2}$ with the decomposition $\Phi = \xi + \theta w$.

The Lax representation (29) gives us the consistent equation only for the special choice of the parameters λ_i . We have five solutions for the coefficients.

1.) For $\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 1$ we obtain B-extension of Kaup - Kupershmidt equation

$$\Phi_t = \Phi_{5x} + 10\Phi_{xxx}\Phi_1 + 15\Phi_{xx}\Phi_{1,x} + 10\Phi_x\Phi_{1,xx} + 20\Phi_x\Phi_1^2 \quad (35)$$

2.) We have B-extension of Sawada - Kotera equation

$$\Phi_t = \Phi_{5x} + 5\Phi_{xxx}\Phi_1 + 5\Phi_x\Phi_{1,xx} + 5\Phi_x\Phi_1^2 \quad (36)$$

for the $\lambda_1 = \lambda_3 = 1, \lambda_2 = 0$ or $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$ or $\lambda_2 = \lambda_3 = 0, \lambda_1 = 1$.

3.) We have $N = 1$ supersymmetric extension of Sawada - Kotera equation considered in [28]

$$\Phi_t = \Phi_{5x} + 5\Phi_{xxx}\Phi_1 + 5\Phi_{xx}\Phi_{1,x} + 5\Phi_x\Phi_1^2 \quad (37)$$

for $\lambda_1 = \lambda_3 = 1, \lambda_2 = -1$. In the components last equation reads

$$\xi_t = \xi_{5x} + 5w\xi_{xxx} + 5w_x\xi_{xx} + 5u^2\xi_x \quad (38)$$

$$w_t = w_{5x} + 5ww_{xxx} + 5w_xw_{xx} + 5w^2w_x - 5\xi_{xxx}\xi_x \quad (39)$$

from which we see that the bosonic sector is modified by the term $\xi_{xxx}\xi_x$ and therefore it is not a B extension.

4 Odd Bi-Hamiltonian structure.

For the B-Extension of the Kaup - Kupershmidt or the Sawada - Kotera equations it is easy to construct conserved currents and a Bi-Hamiltonian structure using the prescription described above. Hence these generalizations are integrable. For the supersymmetric $N = 1$ generalization of the Sawada - Kotera equation the situation is much more complicated. In order to find the Bi - Hamiltonian structure let us first make several observations.

I: The Lax operator (34) does not produce the supersymmetric generalization of the Kaup - Kupershmidt equation. Taking into account that the classical Kaup - Kupershmidt equation, as well as the classical Sawada - Kotera equation, possess the same Hamiltonian operator (see Esq. 31), we tried to apply the supersymmetrized version of this operator (26) in order to obtain supersymmetric generalization of these equations. We verified that this approach gives us some supersymmetric equations whose bosonic limit is reduced to the Sawada - Kotera or to the Kaup - Kupershmidt equation but these equations does not possess the higher order conserved quantities and hence they are not integrable.

II: The Lax operator (34), as it was shown in [28], could be factorized as

$$L = \partial_{xxx} + \Phi_1 \partial - \Phi_x \mathcal{D} + \Phi_{1,x} = \Lambda^2 = (\mathcal{D}^3 + \Phi)^2 \quad (40)$$

The Λ operator belongs to the reduced Manin-Radul supersymmetric KP hierarchy [29]. Due to this factorization we obtain new hierarchy of the supersymmetric equations

$$\Lambda_{t,k} = 9[\Lambda, (\Lambda)_+^{\frac{k}{3}}] \quad (41)$$

where k is a natural number such that $k \neq 3n, 4n, 4n+1, n=0, 1, \dots$

Let us present the first four equations

$$\Phi_{t,2} = \Phi_x, \quad (42)$$

$$\Phi_{\tau,7} = (\Phi_{1,xx} + \frac{1}{2}\Phi_1^2 + 3\Phi\Phi_x)_x \quad (43)$$

$$\Phi_{t,10} = \Phi_{5x} + 5\Phi_{xxx}\Phi_1 + 5\Phi_{xx}\Phi_{1,x} + 5\Phi_x\Phi_1^2 \quad (44)$$

$$\begin{aligned} \Phi_{\tau,11} = & \Phi_{1,5x} + 3\Phi_{1,xxx}\Phi_1 + 6\Phi_{1,xx}\Phi_{1,x} + 2\Phi_{1,x}\Phi_1^2 \\ & - 3\Phi_{4x}\Phi - 2\Phi_{xxx}\Phi_x - 6\Phi_{xx}\Phi\Phi_1 - 6\Phi_x\Phi\Phi_{1,x} \end{aligned} \quad (45)$$

where t denotes usual even time while τ odd time.

Moreover this factorization implies that the usual formula on the conserved quantities, as the residue of $L^{\frac{n}{3}}$ does not lead us to any conserved quantities [28]. On the other side if we apply this formula to Λ operator, we obtain the whole hierarchy of local superfermionic conserved quantities. For example the first three quantities are

$$H_1 = \int dx d\theta \Phi \Phi_x = \int dx \xi_x w \quad (46)$$

$$H_2 = \frac{1}{6} \int dx d\theta (3\Phi\Phi_{xxx} + 2\Phi_1^3) = \int dx (\xi_{xxx}w + \xi_x w^2) \quad (47)$$

$$H_3 = \int dx d\theta \left(\Phi_{7x}\Phi + 8\Phi_{xxx}\Phi\Phi_{1,xx} + \Phi_x\Phi(4\Phi_{1,4x} + 20\Phi_{1,xx}\Phi_1 + 10\Phi_{1,x}^2 + \frac{8}{3}\Phi_1^3) \right) \quad (48)$$

These charges are conserved for the supersymmetric Sawada - Kotera equation and does not reduce in the bosonic limit, where all fermions functions disapeare, to the classical charges. Therefore in general, we can not conclude that the integrability of the supersymmetric models implies the integrability of bosonic sector.

III: Let us notice that the exotic equations (43), (45) and $\Phi_{\tau,19}$ (as we also checked) could be rewritten as

$$\Phi_{\tau,7} = \Pi \Phi_{t,2} = (\mathcal{D}^2 + 2\partial\Phi + 2\Phi\partial + \mathcal{D}\Phi\mathcal{D}) \frac{\delta H_1}{\delta \Phi} \quad (49)$$

$$\Phi_{\tau,11} = \Pi \frac{\delta H_2}{\delta \Phi}, \quad \Phi_{\tau,19} = \Pi \frac{\delta H_3}{\delta \Phi} \quad (50)$$

and hence the usual even supersymmetric second Hamiltonian operator of the supersymmetric Korteweg - de Vries equation see Esq. 26 creates the exotics equations. However this Hamiltonian operator is not our desired Hamiltonian operator which generates the physical equations.

IV: The densities of the conserved quantities (46) - (48) are superbosonic functions and hence their gradients are superfermionic functions. As Φ is a superfermionic function, it forces that the expected Hamiltonian operator, responsible for the creation of physical equations, to be a superboson operator. This conclusion implies that such Hamiltonian operator, if exists, creates an odd Hamiltonian structures with the antibrackets as the Poisson brackets.

This observations allow us to construct an odd Bihamiltonian structure for the $N = 1$ supersymmetric Sawada - Kotera as follows:

First let us notice that the supersymmetric $N = 1$ Sawada - Kotera equation could be rewritten as

$$\Phi_{t,10} = (\mathcal{D}\partial^2 + 2\partial\Phi + 2\Phi\partial + \mathcal{D}\Phi\mathcal{D}) \left(\Phi_{1,xx} + \frac{1}{2}\Phi_1^2 + 3\Phi\Phi_x \right) \quad (51)$$

In this formula we have correct Hamiltonian operator, while the last term is not a gradient of some superfunction. However if we differentiatiate this term, then it becomes seventh flow in our hierarchy (43). So using the formula (49) we arrive to the following theorem

Theorem 1: The operator

$$P = (\mathcal{D}\partial^2 + 2\partial\Phi + 2\Phi\partial + \mathcal{D}\Phi\mathcal{D}) \partial^{-1} (\mathcal{D}\partial^2 + 2\partial\Phi + 2\Phi\partial + \mathcal{D}\Phi\mathcal{D}) = \Pi\partial^{-1}\Pi \quad (52)$$

is a proper Hamiltonian operator which generates the supersymmetric $N = 1$ Sawada - Kotera equation $\Phi_{t,10} = P \frac{\delta H_1}{\delta \Phi}$ and satisfy the Jacobi identity.

Proof. We have to check the following identity [38]

$$\langle \alpha, P'_{P\beta} \gamma \rangle + \langle \beta, P'_{P\gamma} \alpha \rangle + \langle \gamma, P'_{P\alpha} \beta \rangle = 0. \quad (53)$$

where $P'_{P\beta}$ denotes the Gâteaux derivative along the vector $P\beta$ and α, β, γ are the test superfermionic functions. After a lengthly ¹ calculation these formula could be reduced to the form in which the typical term is

$$\int dx d\theta \left(J_0 + J_{1,1} \partial^{-1} J_{1,2} + J_{2,1} \partial^{-1} J_{2,2} \partial^{-1} J_{2,3} + \dots \right) \quad (54)$$

where the expressions $J_0, J_{n,m}$ are constructed out of $\Phi, \alpha, \beta, \gamma$ and their different (susy)derivatives but does not contain the integral operator. Using the rule

$$\alpha_1 = \overleftarrow{\mathcal{D}}\alpha + \alpha\overrightarrow{\mathcal{D}} \quad , \quad \alpha_x = \overleftarrow{\partial}\alpha - \alpha\overrightarrow{\partial} \quad (55)$$

we can eliminate (susy)derivative from the test superfermionic function α then from β and finally from γ . As the result we obtain that the Jacobi identity reduces to the form with the typical terms

$$\int dx d\theta \left(\Phi_n \alpha_m \Phi_k \beta_l \partial^{-1} \Phi_s \gamma_r \pm \Phi_s \gamma_r \partial^{-1} \Phi_n \alpha_m \Phi_k \beta_l + \Phi_{\hat{n}} \alpha_{\hat{m}} \partial^{-1} \Phi_{\hat{k}} \beta_{\hat{l}} \partial^{-1} \Phi_{\hat{s}} \gamma_{\hat{r}} + \dots \right) \quad (56)$$

where the indices can take values 0, 1, 2, 3, 4 and \pm depends on the parity of the superfunction under the integral operator. Due to the antisymmetric property of integral operator ∂^{-1} all terms cancels out and the Jacobi identity holds. •

It that way we obtained the odd Hamiltonian operator which generates the physical equations and as well as the supersymmetric Sawada - Kotera equation. Regarding P operator as a linear map from the co-vector fields to the vector fields the odd Hamiltonian operator can now be decomposed into

$$\frac{d}{dt} \begin{pmatrix} u \\ \xi \end{pmatrix} = \begin{pmatrix} 2(\partial^2 \xi_x + \xi_x \partial^2) + 3\partial \xi_x \partial + 2\xi_x \partial^{-1} w_x - 2w_x \partial^{-1} \xi_x + 4w \xi_x \\ \partial^4 - 2w \partial^{-1} w_x + 4w^2 + \xi_x \partial + \partial \xi_x \\ w \partial^2 + 2\partial(\partial w + w \partial) - 2\xi_x \partial^{-1} \xi_x \end{pmatrix} - \begin{pmatrix} \partial^4 + 2w_x \partial^{-1} w + 4w^2 + \partial^2 w \\ + 2(\partial w + w \partial) \partial - 2\xi_x \partial^{-1} \xi_x \\ - 2\xi_x \partial^{-1} w - 2w \partial^{-1} \xi_x \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta \xi} \end{pmatrix} \quad (57)$$

We tried to verify the validity of the accidentally found Hamiltonian operator (52) using a natural way of associating several Hamiltonians structure to a given Lax operator [32] - [37]. This approach yields multi - Hamiltonian formulations for the isospectral flows connected to the scattering problem given by that Lax operator. Let us briefly presents the main steps of this procedure. The multi- Hamiltonian structure Γ could be recovered from the given Lax operator computing

$$\dot{L} = \Gamma_1 \nabla F = (L(\nabla F))_+ - ((\nabla F)L)_+ \quad (58)$$

$$\dot{L} = \Gamma_2 \nabla F = L((\nabla F)L)_+ - (L(\nabla F))_+ L \quad (59)$$

$$\dot{L} = \Gamma_3 \nabla F = L(L(\nabla F)L)_+ - (L(\nabla F)L)_+ L - L((\nabla F)L)_+ L + L(L(\nabla F))_+ L \quad (60)$$

¹the computations are simplified if one use the computer algebra Reduce [39] and the package Susy2 [40]

where ∇F denotes the gradient of some conserved quantity.

Usually the multi - Hamiltonians Γ_i are called the "linear", the "quadratic" and the "cubic" operators for $i = 1, 2, 3$, respectively. Given a Hamiltonian function $H(L)$, where the Lax operator L may be regarded as element of the algebra of super pseudo-differential operators

$$L := \sum_{k < \infty} (a_k + b_k \mathcal{D}) \partial^k \quad (61)$$

a convenient parametrization of gradient ∇H is

$$\nabla H = \sum_{k \geq 0} \partial^{-k-1} \left(-\mathcal{D} \frac{\delta H}{\delta a_k} + \frac{\delta H}{\delta b_k} \right) \quad (62)$$

In this parametrisation the trace duality has the usual Euclidean form.

Now, trying to evaluate the first Hamiltonian operator Γ_1 for $L = \mathcal{D}\partial + \Phi$, one immediately encounters a technical difficulty: the corresponding Poisson bracket cannot be properly restricted to this Lax operator. Therefore one should first embed this operator into a larger subspace as

$$L = \mathcal{D}\partial + a\partial + \Phi + b\mathcal{D} \quad (63)$$

and assume that

$$\nabla H = \partial^{-1} \left(\frac{\delta H}{\delta b} - \mathcal{D} \frac{\delta H}{\delta \Phi} \right) - \partial^{-2} \mathcal{D} \frac{\delta H}{\delta a} \quad (64)$$

Thus the Hamiltonian equation $\dot{L} = \Gamma_1 \nabla H$ could be transformed to the matrix form

$$\begin{pmatrix} \dot{b} \\ \dot{\Phi} \\ \dot{a} \end{pmatrix} = \begin{pmatrix} 0 & -\mathcal{D} & 0 \\ -\mathcal{D} & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta b} \\ \frac{\delta H}{\delta \Phi} \\ \frac{\delta H}{\delta a} \end{pmatrix} \quad (65)$$

The next step is to restrict this operator to the smaller subspace where our Lax operator lives. It means that we should apply the Dirac reduction technique to the subspace where $a = 0$ and $b = 0$. We have the standard reduction lemma for Poisson brackets [37] which can be formulated as follows.

For the given Poisson tensor

$$P(v, w) = \begin{pmatrix} P_{v,v} & P_{v,w} \\ P_{w,v} & P_{w,w} \end{pmatrix} \quad (66)$$

let us assume that P_{vv} is invertible, then for arbitrary v the map given by

$$\Theta(w : v) = P_{w,w} - P_{w,v} P_{v,v}^{-1} P_{v,w} \quad (67)$$

is a Poisson tensor where v enters as a parameter rather than as a variable.

Unfortunately it is impossible to make such Dirac reduction for the matrix (65). The same situation occurs in the classical case.

The situation changes when we compute quadratic brackets. Then the analogous matrix to the matrix in (65) is

$$\begin{pmatrix} \mathcal{D}\partial + 2\Phi & -\Phi\mathcal{D} & -\partial \\ -\mathcal{D}\Phi & -\mathcal{D}\partial^2 - \partial\Phi - \Phi\partial & \mathcal{D}\partial \\ \partial & -\mathcal{D}\partial & \mathcal{D} \end{pmatrix} \quad (68)$$

Now we can make the Dirac reduction with respect to the last column and last row and we obtain

$$\hat{\Gamma}_1 = \begin{pmatrix} 2(\mathcal{D}\partial + \Phi) & -(\mathcal{D}\partial + \Phi)\mathcal{D} \\ -\mathcal{D}(\mathcal{D}\partial + \Phi) & -\partial\Phi - \Phi\partial \end{pmatrix} \quad (69)$$

Noticing that

$$\hat{\Gamma}_{2,1} \hat{\Gamma}_{1,1}^{-1} \hat{\Gamma}_{1,2} = \frac{1}{2} (\partial^2 \mathcal{D} + \mathcal{D}\Phi\mathcal{D}) \quad (70)$$

we can finally carry out the reduction with respect to the first column and the first row obtaining that the second Hamiltonian operator is proportional to second Hamiltonian operator of the supersymmetric Korteweg de Vries equation $\Gamma_1 = -\frac{1}{2}\Pi$.

The third Hamiltonian operator produces the complicated 6×6 matrix

$$\hat{I}_3 = \begin{pmatrix} \partial\Phi_1 - \Phi_x\mathcal{D} & -\partial\Phi_1\mathcal{D} & 0 & \Phi_x & \partial & -\partial\mathcal{D} \\ \mathcal{D}\Phi_1\partial & -\partial\Phi_1\partial & -\partial^2\mathcal{D} - \mathcal{D}\Phi_1 & \Phi_{1,x} & \partial\mathcal{D} & \Phi_1 \\ 0 & -\mathcal{D}\partial^2 - \Phi_1\mathcal{D} & 2\partial & \mathcal{D}\partial & 0 & -\mathcal{D} \\ -\Phi_x & \Phi_{1,x} & -\mathcal{D}\partial & \Phi_1 & \mathcal{D} & \partial \\ \partial & -\mathcal{D}\partial & 0 & \mathcal{D} & 0 & 0 \\ \mathcal{D}\partial & \Phi_1 & -\mathcal{D} & -\partial & 0 & 2 \end{pmatrix} \quad (71)$$

Again we can invoke Dirac reduction to the subspace spanned by Φ only. We first make the reduction with respect to the fourth, fifth and sixth row and fourth, fifth and sixth column. Taking into account that it is possible to find the inverse matrix to the matrix constructed out of these columns and rows

$$\begin{pmatrix} \Phi_1 & \mathcal{D} & \partial \\ \mathcal{D} & 0 & 0 \\ -\partial & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \mathcal{D}\partial^{-1} & 0 \\ \mathcal{D}\partial^{-1} & -\mathcal{D}\partial^{-1}\Phi - \Phi\mathcal{D}\partial^{-1} - \frac{1}{2}\partial & -\frac{1}{2}\mathcal{D} \\ 0 & \frac{1}{2}\mathcal{D} & \frac{1}{2} \end{pmatrix} \quad (72)$$

we found the reduced matrix in the form

$$\widetilde{\Gamma}_3 = \begin{pmatrix} 2\partial^3 - 2\Phi_x\mathcal{D} + 2\partial\Phi_1 & -\partial^3\mathcal{D} + \Phi_x\partial - \partial\Phi_1\mathcal{D} & 0 \\ \partial^3\mathcal{D} + \partial\Phi_x + \mathcal{D}\Phi_1\partial & -\frac{1}{2}(\partial^4 + 2\partial\Phi_1\partial + \Phi_{1,xx} + \Phi_1^2) & -\frac{1}{2}(\partial^2\mathcal{D} + \mathcal{D}\Phi_1 + \Phi_x) \\ 0 & \frac{1}{2}(\partial^2\mathcal{D} + \Phi_1\mathcal{D} - \Phi_x) & \frac{3}{2}\partial \end{pmatrix} \quad (73)$$

Again we carry out the Dirac reduction for the last column and last row in $\widetilde{\Gamma}_3$ obtaining

$$\widehat{\Gamma}_3 = \begin{pmatrix} 2(\partial^3 - \Phi_x\mathcal{D} + \partial\Phi_1) & -(\partial^3 - \Phi_x\mathcal{D} + \partial\Phi_1)\mathcal{D} \\ \mathcal{D}(\partial^3 - \Phi_x\mathcal{D} + \partial\Phi_1) & \widehat{\Gamma}_{3,2,2} \end{pmatrix} \quad (74)$$

where

$$\widehat{\Gamma}_{3,2,2} = -\frac{1}{3}(2\partial^4 + 4\Phi_1\partial^2 + 3\Phi_{1,x}\partial + 2\Phi_x\mathcal{D}\partial + \Phi_{1,xx} + 2\Phi_1^2 + \Phi_{xx}\mathcal{D} - \mathcal{D}\Phi_1\partial^{-1}\Phi_x + \Phi_x\partial^{-1}\Phi_1\mathcal{D}) \quad (75)$$

It is possible to carry out the last reduction for the first column and first row in $\widehat{\Gamma}_3$ because

$$\widehat{\Gamma}_{32,1}\widehat{\Gamma}_{31,1}^{-1}\widehat{\Gamma}_{31,2} = \frac{1}{2}\mathcal{D}(\partial_{xxx} - \Phi_x\mathcal{D} + \partial\Phi_1)\mathcal{D} \quad (76)$$

and as the result we obtained that the third Hamiltonian operator is proportional to our odd Hamiltonian operator (52) $\Gamma_3 = -\frac{1}{6}P$

As we seen these two Hamiltonian operators produces two different series of equations, the even Γ_2 operator generates the exotic equations, while the odd one Γ_3 generates the supersymmetric Sawada - Kotera equation. So these operators could be considered independently and one can ask wheather these two series possess second Hamiltonian operator. For the odd Hamiltonian operator we can find such by factorization of the recursion operator found in [28] as $R = PJ$. As the result we were able to prove the following theorem

Theorem 2: The J operator defines a proper implectic operator for the $N = 1$ supersymmetric Sawada - Kotera equation

$$J = \partial_{xx} + \Phi_1 - \partial^{-1}\Phi_{1,x} + \partial^{-1}\Phi_x\mathcal{D} + \Phi_x\partial^{-1}\mathcal{D} \quad (77)$$

Proof. The implectic operator should satisfy [38]

$$\langle \alpha, P'_\beta \gamma \rangle + \langle \beta, P'_\gamma \alpha \rangle + \langle \gamma, P'_\alpha \beta \rangle = 0. \quad (78)$$

After computing this expression we follow the same strategy as in the theorem 1 and verify that this is indeed zero. •

This operator generate the gradient of the conserved quantity according to the formula

$$J\Phi_t = \frac{\delta H_3}{\delta \Phi} \quad (79)$$

On the other side the implectic operator could be decomposed in the following manner

$$\begin{pmatrix} \frac{\delta H}{\delta w} \\ \frac{\delta H}{\delta \xi} \end{pmatrix} = \begin{pmatrix} \xi_x\partial^{-1} + \partial^{-1}\xi_x & \partial^2 - \partial^{-1}w_x + w \\ -\partial^2 - w_x\partial^{-1} - w & \xi_x \end{pmatrix} \begin{pmatrix} w_t \\ \xi_t \end{pmatrix} \quad (80)$$

The recursion operator found in [28] does not generate exotic flows in the supersymmetric Sawada - Kotera hierarchy. More precisely we checked that $R\Phi_{\tau,7} \neq \Phi_{\tau,14}$. Unfortunately we were not been able to find any second Hamiltonian or recursion operator for the exotic series.

5 Conclusion

In this paper we found new unusual features of the supersymmetric models, We showed that the supersymmetric extension of the Sawada - Kotera equation has an odd Bi-Hamiltonian structure. The exotic equations in the supersymmetric Sawada- Kotera hierarchy are generated by the same supersymmetric Hamiltonian operator which appeared in the supersymmetric Korteweg de Vries equation. The existence of the Bi-Hamiltonian structure allow us to state that this model is integrable. Unfortunately we did not found any second Hamiltonian operator or recursion operator responsible for exotics equations. It seems reasonable to assume that the second recursion operator should exist which is supported by the observation that in the classical models, without supersymmetry, two different recursion operators could exists [41], [42]. It will be also interesting to find more examples of the supersymmetric models with the similar or the same properties as the supersymmetric Sawada - Kotera equation.

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